Constructing Quantum Mechanics from a Clifford substructure of the relativistic point particle

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Abstract

We show that the quantized free relativistic point particle is equivalent to a classical string in a Clifford Space which generates the space-time coordinates through its inner product. The generating algebra is preserved by a unitary symmetry which becomes the symmetry of the quantum states. We start by resolving the space-time canonical variables of the point particle into inner products of Weyl spinors with components in a Clifford algebra. Next, we build a dynamical system of N particles which has a $U(N)$ symmetry that mixes the coordinates and momenta belonging to different particles. The reverse problem is formulated in terms of the commutation relations $[X, P] = ik$. For $k = 0$, the system can be gauged back into a system of independent particles. For $k \neq 0$, the system becomes an irreducible and infinite system of coupled particles and generates a space-time canonical system formally identical to that of Matrix Mechanics. The continuum limit is identified as a particular parametrization of a relativistic string in Clifford Space.

1 Introduction

There are two reasons why Clifford algebras are interesting for a deeper understanding of the relationship between Quantum Mechanics and space-time structure. The first one is that in even dimensions they come with a built-in unitary symmetry in their generating algebra which can serve as a basis for the unitary symmetry of quantum states. This is the subject matter of this paper. The second reason, as argued in [1], is that Quantum Mechanics can be understood as a local theory on such a non-commutative space. This resolves the apparent paradox that Quantum Mechanics, though formulated as a local theory, shows non-local behavior. We shall briefly elaborate on this second point.

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Unlike other statistical theories, Quantum Mechanics employs probabilities that are not primary quantities, but are expressed in terms of underlying linear complex amplitudes. When applied to experiments like the single particle double slit experiment, this leads to interference terms in the probabilities which signal an apparent non-local behavior. This does not show that Quantum Mechanics *per se* is non-local, only that it is non-local with respect to Space-Time (or any equivalent commutative space). If Space-Time was generated by an underlying space with the structure group $SL(2.C)$, it is not difficult to imagine that the double homomorphism $SL(2.C) \rightrightarrows SO(1.3)$ would permit a local interpretation of the interference terms. For mathematical reasons such a space would necessarily have to be non-commutative and therefore Bell's theorem [2] would not apply. In [1] we studied a model of this kind. The space-time coordinates x^μ of the relativistic point particle were resolved into spinors with components in a Clifford algebra according to

$$\sigma_\mu^{A\dot{B}} x^\mu = c^A \bullet c^{*\dot{B}}$$

where σ are the Pauli matrices, c^A transforms like a two-component Weyl spinor and $\bullet$ is the inner product of the Clifford algebra. The easiest way to understand how this can affect locality is to consider the transition amplitudes in terms of paths in Clifford Space. In [1] we showed that, in a suitable parametrization, the simplest possible classical trajectories for the relativistic point particle are

$$c^A = a^A \tau \,, \quad x^\mu = b^\mu \tau^2 \,, \, \tau \in R \tag{1}$$

The trajectories in Clifford Space cover the trajectories in Space-Time twice with $c(\tau)$ and $c(-\tau)$ corresponding to the same space-time point $x(\tau^2)$. We assumed that this would also apply during a measurement. In the quantum regime, however, paths are not restricted in this way and can contain points corresponding to different positions in space at the same proper time τ^2. When in the double slit experiment a particle travels from one point to another through two slits, there are only two alternative sets of paths in Space-Time but four alternative sets of paths in Clifford Space. The transition amplitude in Clifford Space is the sum of four parts and is identical to the Space-Time transition probability. The two interference terms in the transition probability are simply the amplitudes for the particle to travel along a path in Clifford Space which passes through both slits at opposite values of τ. According to this view, the resolution of the locality problem follows from the topology of the Lorentz group and is implemented by a non-commutative representation.

The Clifford model differs from conventional space-time physics in that the world line of a particle cannot be extended into *both* the infinite past and the infinite future as measured in proper time. In Space-Time, the endpoint of the trajectory would appear as a singular point, but in Clifford Space it merely represents a 'turning point' where the space-time trajectory is being reproduced for the second time.

2 Mathematical preliminaries

Throughout this paper, we shall make use of the connection between real four-vectors and second-rank hermitian spinors

$$V^\mu = \frac{1}{2}\sigma^\mu_{A\dot{B}} V^{A\dot{B}} \,, \quad V^{A\dot{B}} = \sigma^{A\dot{B}}_\mu V^\mu \tag{2}$$

where σ_μ are the four hermitian Pauli matrices.

It is well known that a null vector can be resolved into a product of two Weyl spinors

$$x^{A\dot{B}} = c^A \cdot c^{*\dot{B}} \,, \quad x^\mu x_\mu = 0 \tag{3}$$

To resolve non-null vectors, we need something like

$$x^{A\dot{B}} = c^A \bullet c^{*\dot{B}} \tag{4}$$

where $\bullet$ is a product which belongs to some non-commutative algebra. This problem can be compared to the somewhat similar problem of resolving the Lorentz metric $\eta_{\mu\nu}$ into vectors. The well known solution is $\eta_{\mu\nu} = \frac{1}{2}\{\gamma_\mu, \gamma_\nu\}$ where the Dirac matrices γ_μ generate the Clifford algebra $\mathcal{C}l(1, 3, \mathbb{R})$. The components of any real symmetric 4×4 matrix of signature $(1, 3)$ can therefore be expressed as the inner products (anti-commutators) of vectors (real linear combinations of γ matrices) belonging to $\mathcal{C}l(1, 3, \mathbb{R})$. Real Clifford algebras are associated with real quadratic forms, but there is no similar connection between hermitian sesquilinear forms and complex Clifford algebras $\mathcal{C}l(\mathbb{C})$. Instead we must use even-dimensional real Clifford algebras written in complex form. Consider a future directed time-like vector x^μ. A unitary transformation followed by a non-uniform scaling can reduce $x^{A\dot{B}}$ to a diagonal matrix with ones in the diagonal and can be effected by a suitable linear transformation of c^A so that (4) becomes

$$c_i \bullet c^*_j = \delta_{ij} \tag{5}$$

This can be compared to the algebra of creation and annihilation operators for two fermions

$$\{\mathbf{a}_i, \mathbf{a}^\dagger_i\} = \delta_{ij} \cdot 1 \,, \quad \{\mathbf{a}_i, \mathbf{a}_j\} = 0 \,, \; i, j = 1, 2 \tag{6}$$

Defining $e_i = i(\mathbf{a}_i + \mathbf{a}^\dagger_i) \,, \; e_{2+i} = \mathbf{a}_i - \mathbf{a}^\dagger_i \,, \; i = 1, 2$, the commutation relations (6) become

$$\{e_i, e_j\} = -2\delta_{ij} \,, \; i, j = 1, \ldots, 4 \tag{7}$$

which generate the Clifford algebra $\mathcal{C}l(0, 4, R)$. This suggests that a solution to (4) would be to use spinors with values in $\mathcal{C}l(4, 4, \mathbb{R})$ and to let $\bullet$ be the inner product (anti-commutator) of this algebra. Since any null vector can be created by letting c contain only a single generator (giving (3)), we don't need to extend $\mathcal{C}l(4, 4, \mathbb{R})$ to a degenerate algebra. This expectation is borne out by the following proposition

Let V_C be a $2n$-dimensional complex linear space with complex conjugation $*$ and H an $n \times n$ Hermitian matrix of arbitrary signature. Then the components of H can be expressed as

$$H_{ij} = c_i \bullet c_j^* \,, \quad c_i \bullet c_j = 0 \,, \, i,j = 1, \dots, n$$

where c_i belong to V_C and $\bullet$ is the inner product

$$a \bullet b \equiv \frac{1}{2}\{a, b\}$$

of the Clifford algebra $\mathcal{C}l(2n, 2n, \mathbb{R})$ on the $4n$ dimensional real linear space V_R which corresponds to V_C .

Proof. Let $e_i, \ f_i, \ i = 1, \dots, n$ be a basis for V_C and $g_i = i(e_i + e_i^*), \ g_{n+i} = e_i - e_i^*, \ h_i = i(f_i + f_i^*), \ h_{n+i} = f_i - f_i^*, \, i = 1, \dots n$ a basis for V_R . Let g_i and h_i generate the Clifford algebra $\mathcal{C}l(2n, 2n, R)$ on V_R through

$$g_i \bullet g_j = 2\delta_{ij} \,, \ h_i \bullet h_j = -2\delta_{ij} \,, \ \ g_i \bullet h_j = 0 \,, \, i,j = 1, \dots, 2n \tag{8}$$

Then the basis $e_i, \ f_i$ for V_C satisfies

$$e_i \bullet e_j^* = -\delta_{ij} \,, \ f_i \bullet f_j^* = \delta_{ij} \,, \ \ e_i \bullet e_j = f_i \bullet f_j = 0 \,, \, i,j = 1, \dots, n \tag{9}$$

We can create any $n \times n$ diagonal matrix with diagonal entries ± 1 by setting c_i equal to either e_i or f_i . A zero in the k-th entry of the diagonal can be created by $c_k = e_k + f_k$. A non-uniform scaling followed by a unitary transformation can transform this diagonal matrix into any desired $n \times n$ hermitian matrix with the same signature and can be effected by a suitable complex linear transformation of the c's $\square$

We shall resolve both the coordinates and momenta of the point particle into Clifford spinors

$$x^{A\dot{B}} = c^A \bullet c^{*\dot{B}} \,, \ \ p_{A\dot{B}} = d_A^* \bullet d_{\dot{B}} \tag{10}$$

but we also need the Clifford algebra to be large enough that the inner products

$$c^A \bullet d_B^* \tag{11}$$

are algebraically independent of x and p. This can for example be accomplished by enlarging $\mathcal{C}l(4, 4, \mathbb{R})$ to $\mathcal{C}l(8, 8, \mathbb{R})$ and then generating x and p by each their own $\mathcal{C}l(4, 4, \mathbb{R})$ subalgebra. This makes $c \bullet d$ vanish. The second step is to choose two Clifford elements h_i whose inner products with both c and d vanish, and to make the substitution

$$c^A \to c^A + A_i^A h_i \,, \ \ d_A^* \to d_A^* + B_{iA} h_i^* \tag{12}$$

This will only change x and p by additive matrices that will not constrain them, and the two matrices A and B can be adjusted to produce any desired value

of the inner product (11). The exact dimension and signature of the single-particle Clifford algebra is not of any importance in this paper. It only needs to be large enough that all inner products between the Clifford canonical variables are algebraically independent of each other.

Note that $c^{*\dot{A}}$ and d^*_A have the same commutation properties but transform differently under $SL(2.\mathbb{C})$. The complex conjugation symbol $*$ can therefore not be omitted, as it often is, because it specifies the commutation properties of the element in question. It is tacitly assumed that the inner product of elements of the same kind vanishes, and this will not be written out explicitly.

The variation of a real function f which depends on c^A only through an inner product can be expressed on the form

$$\delta f = \frac{\partial f}{\partial c^A} \bullet \delta c^A + \frac{\partial f}{\partial c^{*\dot{B}}} \bullet \delta c^{*\dot{B}} \tag{13}$$

which defines the 'derivative' of f with respect to c . This will serve as a convenient notation.

3 Clifford substructure of the relativistic point particle

Let the space-time coordinates and momenta of the relativistic point particle be resolved into Clifford spinors according to (10). The equations of motion are obtained from the condition that the reparametrization invariant action

$$I = 4\sqrt{m}\int \sqrt[4]{\frac{1}{2}\frac{dc^A}{d\tau} \bullet \frac{dc^{*\dot{B}}}{d\tau}\,\frac{dc_A}{d\tau} \bullet \frac{dc^*_{\dot{B}}}{d\tau}}\,d\tau \tag{14}$$

is stationary under arbitrary variations of $c(\tau)$. The momenta conjugate to c are

$$d^*_A \equiv \frac{\partial L}{\partial \frac{dc^A}{d\tau}} = \sqrt{m}\,(\frac{1}{2}\frac{dc^E}{d\tau} \bullet \frac{dc^{*\dot{F}}}{d\tau}\,\frac{dc_E}{d\tau} \bullet \frac{dc^*_{\dot{F}}}{d\tau})^{-\frac{3}{4}}(\frac{dc_A}{d\tau} \bullet \frac{dc^*_{\dot{B}}}{d\tau})\,\frac{dc^{*\dot{B}}}{d\tau} \tag{15}$$

and as expected, the Hamiltonian $d^*_A \bullet \frac{d}{d\tau}c^A + c.c. - L$ vanishes. A straightforward calculation using the four-vector rule

$$V_{A\dot{E}}V^{B\dot{E}} = \frac{1}{2}\delta^B_A V_{F\dot{E}}V^{F\dot{E}} \tag{16}$$

shows that the conjugate momenta d^*_A satisfy the constraint

$$p^\mu p_\mu - m^2 = 0 \tag{17}$$

where p_μ are the space-time momenta defined in (10). This happens to be the same constraint as would have been obtained from the space-time action

$\int \sqrt{\dot{x}^2}\, d\tau$. According to constrained dynamics, the Hamiltonian is proportional to the constraint

$$H(p, e(\tau)) = e(\tau)(p^\mu p_\mu - m^2) \tag{18}$$

where $e(\tau)$ is an einbein. This Hamiltonian can also be obtained from the Polyakov action

$$\int 3e(\tau)^{-\frac{1}{3}} \sqrt[3]{\frac{1}{2}\frac{dc^A}{d\tau} \bullet \frac{dc^{*\dot{B}}}{d\tau}\, \frac{dc_A}{d\tau} \bullet \frac{dc^*_{\dot{B}}}{d\tau}} + m^2\, e(\tau)\, d\tau \tag{19}$$

which recovers (14) when the equations of motion for $e(\tau)$ are substituted back into the action. The momenta conjugate to c, are

$$d^*_A = e(\tau)^{-\frac{1}{3}} (\frac{1}{2}\frac{dc^E}{d\tau} \bullet \frac{dc^{*\dot{F}}}{d\tau}\, \frac{dc_E}{d\tau} \bullet \frac{dc^*_{\dot{F}}}{d\tau})^{-\frac{2}{3}} \frac{dc^{*\dot{B}}}{d\tau}\, \frac{dc_A}{d\tau} \bullet \frac{dc^*_{\dot{B}}}{d\tau} \tag{20}$$

which can be used to determine the Hamiltonian density

$$H(c, d) = d^*_A \bullet \frac{dc^A}{d\tau} + c.c. - L \tag{21}$$

where $c.c.$ denotes the complex conjugate of the previous term and L is the Lagrangian in (19). A straightforward calculation gives

$$d^*_A \bullet \frac{dc^A}{d\tau} + c.c. = 4e(\tau)^{-\frac{1}{3}} \sqrt[3]{\frac{1}{2}\frac{dc^A}{d\tau} \bullet \frac{dc^{*\dot{B}}}{d\tau}\, \frac{dc_A}{d\tau} \bullet \frac{dc^*_{\dot{B}}}{d\tau}} \tag{22}$$

$$p^\mu p_\mu \equiv \frac{1}{2} d^A \bullet d^{*\dot{B}}\, d_A \bullet d^*_{\dot{B}} = e(\tau)^{-\frac{4}{3}} \sqrt[3]{\frac{1}{2}\frac{dc^A}{d\tau} \bullet \frac{dc^{*\dot{B}}}{d\tau}\, \frac{dc_A}{d\tau} \bullet \frac{dc^*_{\dot{B}}}{d\tau}} \tag{23}$$

which, when applied to (21), yields the Hamiltonian (18) of constrained dynamics. Hence the first order (Hamiltonian) form of the action (14) is

$$I = \int d^*_A \bullet \frac{dc^A}{d\tau} + c.c. - H(p, e(\tau))\, d\tau \tag{24}$$

This action has a global $SL(2.\mathbb{C})$ and $U(1)$ gauge symmetry with the conserved Noether charges

$$\mathcal{J}_{\mathcal{AB}} \equiv d^*_A \bullet c_B + d^*_B \bullet c_A\,, \quad \jmath \equiv i(d^*_A \bullet c^A - c.c.) \tag{25}$$

To obtain the correct space-time equations of motion, it is necessary to assume (as an initial value condition) that they vanish

$$d^*_A \bullet c_B + d^*_B \bullet c_A = 0 \tag{26}$$

$$d^*_A \bullet c^A - c.c. = 0 \tag{27}$$

Since all skew symmetric second rank tensors are proportional to ϵ_{AB}, (26) gives

$$d^*_A \bullet c_B = \mu(\tau)\,\epsilon_{AB} \tag{28}$$

with (27) saying that $\mu(\tau)$ is real. We shall refer to this condition as the 'Noether condition'. The canonical equations of motion are obtained by independent variation of c and d

$$\frac{dc^A}{d\tau} = \frac{\partial H}{\partial d^*_A} = \frac{\partial H}{\partial p_{A\dot{E}}} d_{\dot{E}}\,, \quad \frac{d\,d^*_A}{d\tau} = -\frac{\partial H}{\partial c^A} = -\frac{\partial H}{\partial x^{A\dot{E}}} c^{*\dot{E}}\,(=0) \tag{29}$$

Taking the inner product of these equations with $c^{*\dot{B}}$ and $d_{\dot{B}}$ gives

$$\frac{dx^{A\dot{B}}}{d\tau} = 2\frac{\partial H}{\partial p_{A\dot{E}}}\, c^{*\dot{B}} \bullet d_{\dot{E}}\,, \quad \frac{dp_{A\dot{B}}}{d\tau} = -2\frac{\partial H}{\partial x^{A\dot{E}}}\, c^{*\dot{E}} \bullet d_{\dot{B}}\,(=0) \tag{30}$$

which by use of the 'Noether condition' (28) become

$$\frac{dx^{A\dot{B}}}{d\tau} = 2\frac{\partial H}{\partial p_{A\dot{B}}}\mu(\tau)\,, \quad \frac{dp_{A\dot{B}}}{d\tau} = -2\frac{\partial H}{\partial x^{A\dot{B}}}\,\mu(\tau)\,(=0) \tag{31}$$

In the parametrization

$$\overline{e}(\overline{\tau}) = \frac{1}{2m\mu(\overline{\tau})} \tag{32}$$

these equations reduce to the canonical equations of motion

$$\frac{dx^\mu}{d\overline{\tau}} = \frac{\partial \mathcal{H}}{\partial p_\mu}\,, \quad \frac{dp_\mu}{d\overline{\tau}} = -\frac{\partial \mathcal{H}}{\partial x^\mu}\,(=0)\,, \quad \mathcal{H}(x,p) \equiv \frac{1}{2m}(p^\mu p_\mu - m^2) \tag{33}$$

for a relativistic point particle with proper time $\overline{\tau}$. This proper time is not defined at points where μ vanishes. There will be just one such point and it represents a 'turning point' where the space-time trajectory has an endpoint and the underlying trajectory in Clifford Space starts to reproduce it for the second time. From (29) and the Hamiltonian constraint (17), we obtain an explicit expression for $\mu(\tau)$

$$\frac{d}{d\tau}\mu(\tau) = \frac{d}{d\tau}(\frac{1}{2}d^*_E \bullet c^E) = e(\tau)m^2\,, \quad \mu(\tau) = \int_{\tau_0}^{\tau} m^2 e(t)\,dt \tag{34}$$

Hence $\mu(\tau)$ is determined by the mass of the particle and the turning point τ_0 of its motion.

4 System of N particles with a U(N) symmetry

If the Clifford algebra for a single particle is $Cl(r,s,\mathbb{R})$, then N particles can be accommodated in $Cl(Nr,Ns,\mathbb{R})$ in such a way that all inner products between

Clifford coordinates and momenta belonging to different particles vanish. The generating algebra

$$e_i^p \bullet e_j^{*q} = \delta_{ij}\delta_{pq}\, sign(p)\,, \quad e_i^p \bullet e_j^q = 0\,,\ i,j = 1,\dots N\,,\ \ p,q = 1,\dots r+s \quad (35)$$

is preserved by the $U(N)$ unitary transformation

$$e_i^p \to U_{ih} e_h^p\,, \quad U_{ih} U_{jh}^* = \delta_{ij} \quad (36)$$

where $sign(p)$ denotes the sign of $e^p \bullet e^{*p}$ in the generating algebra of $Cl(r,s,\mathbb{R})$. If we assemble the canonical variables c_i^A and d_{iA}^*, $i = 1\dots, N$ of the N particles into the ket- and bra-vectors $\overset{>}{C}{}^A$ and $\overset{<}{D}_A$ respectively, then the corresponding space-time coordinates and momenta are elements of the $N \times N$ diagonal matrices

$$X^{A\dot{B}} = \overset{>}{C}{}^A \bullet \overset{<}{C}{}^{\dot{B}}, \quad P_{A\dot{B}} = \overset{>}{D}_{\dot{B}} \bullet \overset{<}{D}_A \quad (37)$$

which trivially satisfy the commutation relations

$$[X^\mu, X^\nu] = [P_\mu, P_\nu] = [X^\mu, P_\nu] = 0 \quad (38)$$

The equations of motion for the N particles can be derived from the sum of the single-particle actions (24). When the particles have the same mass, this action can be written as

$$I = \int Tr\,(\frac{d}{d\tau}\, \overset{>}{C}{}^A \bullet \overset{<}{D}_A + c.c. - H\,)\, d\tau\,, \quad H \equiv e(\tau)(P^\mu P_\mu - m^2 \cdot \underline{1}) \quad (39)$$

Assuming that the particles also have the same turning points and thereby the same $\mu(\tau)$, the 'Noether condition' (28) becomes

$$\overset{>}{C}{}^A \bullet \overset{<}{D}_B = \mu(\tau)\, \delta_B^A \cdot \underline{1} \quad (40)$$

We observe that this N-particle system is preserved by the global $U(N)$ transformations

$$\overset{>}{C}{}^A \to U\ \overset{>}{C}{}^A, \quad \overset{<}{D}_A \to \overset{<}{D}_A\ U^\dagger \quad (41)$$

which produce the similarity transformations

$$X^\mu \to U X^\mu U^\dagger\,, \quad P_\mu \to U P_\mu U^\dagger \quad (42)$$

of the Hermitian matrices X^μ and P_μ. Such transformations create off-diagonal entries in X and P which correspond to artificial couplings between Clifford coordinates and momenta belonging to different particles.

5 Matrix Mechanics

We have seen that N independent particles in Clifford Space leads to a unitarily invariant dynamical system. The reverse problem is to determine under which conditions such a type of system can be gauged back into a set of independent particles. To address this problem, we must define a unitarily invariant system which relaxes one or more assumptions associated with independence. A natural starting point is to define an action principle for the whole system which is equivalent to the combined action principles for the particles it contains. To this end, we observe that to make N single-particle actions stationary is equivalent to making all time-independent linear combinations of them stationary. This corresponds to the action

$$I = \int \sum_{i=1}^{N} \phi_i L_i \, d\tau \,, \quad L_i = d^*_{iA} \bullet \frac{dc_i^A}{d\tau} + c.c. - H(p_i, e(\tau)) \tag{43}$$

where the coefficients ϕ_i are arbitrary real constants, and L_i are the single-particle Lagrangians. In section 7, the ϕ's will be given a geometrical interpretation. When Φ denotes the $N \times N$ diagonal matrix with ϕ_i along its diagonal, the action (43) can be written as

$$I = \int Tr \left(\Phi(\frac{d}{d\tau} \overset{>}{C}{}^A \bullet \overset{<}{D}_A + h.c. - H) \right) d\tau \tag{44}$$

where H is the Hamiltonian in (39) with P being diagonal. This action is preserved by the unitary transformation (41) with Φ transforming according to

$$\overline{\Phi} \to U \Phi U^\dagger \,, \quad \Phi^\dagger = \Phi \tag{45}$$

It will prove convenient to express this system in a 'picture'-independent form by turning the global $U(N)$ symmetry into a local one. This is done by means of an auxiliary $U(N)$ gauge connection Γ

$$\Gamma \to U \Gamma U^\dagger - i \frac{dU}{d\tau} U^\dagger \,, \quad \bar{\Gamma}(\bar{\tau}) = \Gamma(\tau) \frac{d\tau}{d\bar{\tau}} \tag{46}$$

which defines the gauge covariant derivative

$$\nabla_\tau \overset{>}{V}{}^A \equiv \frac{d}{d\tau} \overset{>}{V}{}^A - i\Gamma(\tau) \overset{>}{V}{}^A, \quad \bar{\nabla}_{\bar{\tau}} \overset{>}{V}{}^A \equiv \frac{d}{d\bar{\tau}} \overset{>}{V}{}^A - i\bar{\Gamma}(\bar{\tau}) \overset{>}{V}{}^A \tag{47}$$

The local $U(N)$ invariant form of the action (44) is then

$$I = \int Tr \left(\Phi(\nabla_\tau \overset{>}{C}{}^A \bullet \overset{<}{D}_A + h.c. - H) \right) d\tau \tag{48}$$

In $1+0$ dimensions, Γ is not a dynamical field and we assume that it vanishes in the gauge where Φ and P are diagonal. The diagonal matrices Φ and P trivially satisfy the gauge invariant conditions

$$\nabla_\tau \Phi \equiv \frac{d}{d\tau}\Phi - i[\Gamma, \Phi] = 0 \tag{49}$$

$$[\Phi, P_\mu] = 0 \tag{50}$$

$$[P_\mu, P_\nu] = 0 \tag{51}$$

Conversely, these conditions ensure that the action (48) can be gauged back into (43).

Equations (48)-(51) describe a general class of unitarily invariant dynamical systems which includes, but is not limited to, systems of independent particles. Systems of independent particles are obtained by adding the commutation relations

$$[X^\mu, X^\nu] = 0 \tag{52}$$

$$[X^\mu, P_\nu] = 0 \tag{53}$$

which ensure that all off-diagonal entries of X and P, that is all couplings between different particles, can be gauged away in the same unitary frame. A larger family of systems is obtained by replacing (53) with

$$[X^\mu, P_\nu] = ik\delta^\mu_\nu \tag{54}$$

which is also preserved by the equations of motion. For $k \neq 0$, the couplings between coordinates and momenta can no longer be gauged away and we obtain an irreducible and infinite system of coupled particles. When X and P satisfy (54) and hence are conjugate canonical variables, X diverges when P approaches diagonality. This does not affect the action (48), it being independent of X. The conserved $SL(2.\mathbb{C})$ and $U(N)$ Noether charges are assumed to vanish for all values of Φ. Proceeding in the same manner as in section 3, this gives

$$\overset{>}{C}{}^A \bullet \overset{<}{D}_B = M(\tau)\, \delta^A_B \tag{55}$$

where $M(\tau)$ is a Hermitian matrix. The assumption that all particles have the same mass and turning points is replaced with the assumption that M's eigenvalues are all equal. This reduces (55) to (40).

The equations of motion are obtained by requiring the action (48) to be stationary for all Φ which satisfy (49). By independent variation of C and D and using (49), we obtain

$$\nabla_\tau \overset{>}{C}{}^A = \frac{\partial H}{\partial P_{A\dot{E}}} \overset{>}{D}_{\dot{E}}\,, \quad \nabla_\tau \overset{<}{D}_A = -\overset{<}{C}{}^{\dot{E}} \frac{\partial H}{\partial X^{A\dot{E}}}\ (=0) \tag{56}$$

Taking the inner product on both sides of these equations with $\overset{<}{C}{}^{\dot{B}}$ and $\overset{>}{D}_{\dot{B}}$ and applying (40) and the reparametrization (32), we obtain

$$\overline{\nabla}_{\overline{\tau}} X^\mu = \frac{\partial \mathcal{H}}{\partial P_\mu}\,, \quad \overline{\nabla}_{\overline{\tau}} P_\mu = -\frac{\partial \mathcal{H}}{\partial X^\mu}\,(=0) \tag{57}$$

$$\mathcal{H} \equiv \frac{1}{2m}(P^\mu P_\mu - m^2 \cdot \underline{1}) \tag{58}$$

For $k \neq 0$, the commutation relations (54) allow the derivatives of $\mathcal{H}$ to be written as commutators, which turn these equations into

$$\overline{\nabla}_{\overline{\tau}} X^\mu = \frac{i}{k}[\mathcal{H}, X^\mu]\,, \quad \overline{\nabla}_{\overline{\tau}} P_\mu = \frac{i}{k}[\mathcal{H}, P_\mu]\,(=0) \tag{59}$$

In the gauge $\Gamma = 0$, these equations together with the commutation relations (51),(52) and (54) are formally identical to Matrix Mechanics in the Heisenberg picture. In the gauge $\overline{\Gamma}(\overline{\tau}) = -\frac{1}{k}\mathcal{H}$, the commutators on the left and right hand sides of (59) cancel out and X and P become stationary. This gauge therefore corresponds to the Schrödinger picture.

Note that in the classical system $k = 0$, both the number of trajectories and the initial values of the canonical variables are arbitrary and have to be put in by hand, whereas in the non-classical system the canonical commutation relations (54) determine both the number of trajectories (as being infinite) and automatically provide an infinite range of (stationary) eigenvalues.

6 The state vector

The multi-particle system constructed in the foregoing provides a number of trajectories in Clifford phase space which correspond to a variety of initial values of the canonical variables. Let us consider the classical system $k = 0$. In the gauge where X is diagonal, all trajectories are decoupled from each other and we expect that when, for example, the space-time position x of a particle is being measured at some time τ, a good measurement will return a value $x_i(\tau)$ belonging to one of these trajectories. Expressed in a gauge invariant manner, this is equivalent to saying that it will return an eigenvalue of $X(\tau)$. The result of a measurement can be represented as a gauge invariant expectation value E in terms of a state vector $\mid s >$

$$E(\overset{>}{C}{}^A) \equiv < s \mid \overset{>}{C}{}^A\,, \quad E(X^{A\dot{B}}) \equiv E(\overset{>}{C}{}^A) \bullet E(\overset{<}{C}{}^{\dot{B}}) = < s|X^{A\dot{B}}|s > \tag{60}$$

$\overset{>}{C}(\tau)$ can be expanded in terms of the Clifford coordinates $c_i(\tau)$ which generate the eigenvalues of X

$$\overset{>}{C}{}^A\,(\tau) = \sum_r \mid x_r(\tau)\rangle c_r^A(\tau)\,, \quad c_r^A(\tau) \bullet c_s^{*\dot{B}}(\tau) = \delta_{rs} x_s^{A\dot{B}}(\tau) \tag{61}$$

where $|\, x_i(\tau)\rangle$ denotes the eigenvectors of $X^\mu(\tau)$ with eigenvalues $x_i^\mu(\tau)$. For short, we shall also refer to $c_i(\tau)$ as 'eigenvalues'. It follows that, if the expectation value $E(\overset{>}{C})$ is going to return the correct value c_i of a measurement, the state vector $|\, s >$ must be set equal to the eigenvector $|\, x_i >$. Conversely, if the expectation value coincides with an eigenvalue of X, we would expect a good measurement to return this value. To serve the purpose of predicting the outcome of future measurements, the state vector must be subject to a time evolution. In classical dynamics it is natural to assume that after a measurement has been performed, the expectation value must stay on the trajectory corresponding to this measurement. According to the foregoing, the classical system can be gauged into a set of decoupled trajectories with $X(\tau)$ being diagonal and $\Gamma = 0$. In this gauge the eigenvectors $|\, x_i >$ can be chosen to be constants of motion and the state vector must therefore also be a constant of motion. This leads to the gauge invariant time evolution

$$\nabla_\tau |s > \equiv (\frac{d}{d\tau} - i\Gamma)\, |s >= 0 \tag{62}$$

Incidentally, this time-evolution preserves the norm $< s\,|\, s >$ of the state vector. For the classical system, these measurement principles merely represent a different way of formulating the traditional initial value problem. They do, however, also apply to the non-classical system, irrespective of the fact that the way they were derived is no longer valid. In the non-classical system where X and P do not commute, the assumption that measurements must be expressed through a state vector, imposes restrictions on which type of measurements that can be performed. The time evolution (62) also holds true, as follows from the fact that the state vector is known to be a constant of motion in the Heisenberg picture $\Gamma = 0$. To help appreciate the difference between the classical and the non-classical systems, we expand the expectation value $E(C)$ in terms of the 'eigenvalues' c_i

$$E(\overset{>}{C^A}(\tau)) \equiv < s\,|\, \overset{>}{C^A}(\tau) = < s\,|\, x_i(\tau) > c_i(\tau) \tag{63}$$

In the classical system, in the gauge $\Gamma = 0$, both $< s\,|$ and $|\, x_i >$ are constants of motion and hence the expectation value $E(C(\tau))$ is equal to one of the 'eigenvalues' $c_i(\tau)$. The outcome of a measurement is therefore predictable. This is not surprising since it was used to derive the time evolution of the state vector. In the non-classical system, in the gauge $\Gamma = 0$, the state vector is also stationary, but the eigenvectors $|\, x_i >$ undergo a unitary time evolution. After a measurement has been performed, the expectation value therefore drifts into a complex linear combination of different 'eigenvalues' $c_i(\tau)$. Accordingly, the outcome of a measurement is no longer predictable, but instead occurs with statistical frequencies given by the Born rule.

The classical and non-classical systems are related through Ehrenfest's theorem. The time evolution of the expectation value $E(C)$ is

$$\frac{d}{d\tau}E(\overset{>}{C^A}) = (\nabla_\tau < s|\,)\overset{>}{C^A} + < s|\,\nabla_\tau\overset{>}{C^A} = \langle s|\,\frac{i}{2k}[H, X^{A\dot{E}}]\,\overset{>}{D_{\dot{E}}} \tag{64}$$

Taking the inner product of this equation with $E(\overset{<}{C^{\dot{B}}})$ and using (40) and (60), we obtain after a reparametrization the time evolution of the expectation value of the space-time coordinates

$$\frac{d}{d\overline{\tau}}E(X^\mu) = \langle s\,|\,\frac{i}{k}[\mathcal{H}, X^\mu]\,|\,s\rangle \tag{65}$$

This is Ehrenfest's theorem which could also have been obtained directly from (59) by use of (60) and (62). Note that this derivation is explicit gauge invariant and therefore applies to both the Heisenberg picture $\Gamma = 0$ and the Schrödinger picture $\overline{\Gamma}(\overline{\tau}) = -\frac{1}{k}\mathcal{H}$.

In the non-relativistic limit, the proper time $\overline{\tau}$ is equal to the expectation value of X^0 which represents the physical time $t \equiv < s|X^0|s >$

$$\frac{dt}{d\overline{\tau}} = < s|\bar{\nabla}_{\overline{\tau}}X^0|s > = \frac{1}{m} < s|P^0|s > \approx 1 \tag{66}$$

where we have used the time evolution of the state vector and the equations of motion for X^0. Restricting the equations of motion (59) to $\mu = 1, 2, 3$, the Hamiltonian $\mathcal{H}$ effectively reduces to the non-relativistic Hamiltonian

$$\tilde{H} = \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) \tag{67}$$

Taken together with the corresponding commutation relations, this system is identical to that of non-relativistic Matrix Mechanics. The Schrödinger picture corresponds to the non-relativistic gauge condition $\overline{\Gamma}(t) = -\frac{1}{k}\tilde{H}$ which turns the time evolution (62) of the state vector into the matrix form of the Schrödinger equation.

7 The relativistic string in Clifford Space

To obtain the continuum limit of the multi-particle system constructed in section 5, we start by mapping the generators e_i^p of the generating algebra (35) for $N = \infty$ into the Clifford elements

$$f^p(\sigma) = \sum_{i=1}^{\infty} g_i(\sigma)e_i^p \tag{68}$$

where $g_i(\sigma)$ are complex functions of a real parameter σ. These functions are chosen so that f satisfies

$$f^p(\sigma) \bullet f^{*q}(\sigma') = \delta_n(\sigma - \sigma')\,sign(p)\delta^{pq}\,, \quad f(\sigma) \bullet f(\sigma') = 0 \tag{69}$$

where $\delta_n(\sigma)$, $n = 1, \ldots$ is a sequence of positive even functions which converges to the Dirac delta function $\delta(\sigma)$ for $n \to \infty$. $f^p(\sigma)$ can be regarded as a ket-vector $\overset{>}{f^p}$ with a continuous index σ, and correspondingly $\delta_n(\sigma - \sigma')$ can be regarded as a real symmetric matrix δ_n with continuous indices σ and σ'. For such vectors and matrices with continuous indices, we shall when convenient use the abstract notation of section 4 with summation replaced by integration. The algebra (69) is preserved by the 'pseudo-unitary' transformations

$$\overset{>}{f^p} \to U \overset{>}{f^p} \tag{70}$$

which preserve the metric δ_n

$$U \delta_n U^\dagger = \delta_n \tag{71}$$

In the continuum limit $n \to \infty$ these pseudo-unitary transformations become unitary transformations. The Clifford coordinates $c(\tau, \sigma)$ are defined as integral transforms of f

$$c^A(\tau, \sigma) \equiv \int a_p^A(\tau, \sigma, \sigma') f^p(\sigma')\, d\sigma' \tag{72}$$

and represent a string in Clifford space. The dynamics of this string will be derived from an action principle which is preserved by arbitrary reparametrizations $(\tau, \sigma) \to (\tau', \sigma')$.

It is well known that for a string which resides in Space-Time, the Lorentz metric $\eta_{\mu\nu}$ induces a metric on the worldsheet through the tangent derivatives $\partial_\alpha x^\mu$. For a string which resides in Clifford Space, we use the complex vectors

$$V_\alpha^\mu \equiv \sigma^\mu_{A\dot{B}} c^A \bullet \partial_\alpha c^{*\dot{B}} \tag{73}$$

which have the real part $\partial_\alpha x^\mu$. These vectors induce the hermitian tensor

$$g_{\alpha\beta} \equiv V_\alpha^\mu V_\beta^{\nu *} \eta_{\mu\nu}\,, \quad g^*_{\alpha\beta} = g_{\beta\alpha} \tag{74}$$

on the Clifford worldsheet, which can be decomposed into a real symmetric tensor $h_{\alpha\beta}$ and a real scalar ϕ

$$g_{\alpha\beta} = h_{\alpha\beta} + i\sqrt{h}\phi\epsilon_{\alpha\beta}\,, \quad h_{\alpha\beta} \equiv g_{(\alpha\beta)}\,, \quad \phi \equiv h^{-\frac{1}{2}}\epsilon^{\alpha\beta} g_{\alpha\beta}\,, \quad h \equiv det(h_{\alpha\beta}) \tag{75}$$

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma\,, \quad g \equiv det(g_{\alpha\beta}) = h(1 - \phi^2) \tag{76}$$

The reparametrization invariant string generalization of the Polyakov point particle action (19), is

$$3 \int \sqrt[3]{\frac{1}{2} g^{\alpha\beta} \partial_\alpha c^A \bullet \partial_\beta c^{*\dot{B}}\, g^{\gamma\delta} \partial_\gamma c_A \bullet \partial_\delta c^*_{\dot{B}}}\, \sqrt{g}\, d\tau d\sigma \tag{77}$$

where $g^{\alpha\beta}$ is an inherent hermitian tensor field on the string worldsheet. To express this action in an explicit covariant first order form, we use Dedonder-Weyl covariant canonical variables. It makes the calculations more transparent to define the real four-vector

$$W^\mu \equiv \frac{1}{2}\sigma^\mu_{A\dot{B}} g^{\alpha\beta}\partial_\alpha c^A \bullet \partial_\beta c^{*\dot{B}} \tag{78}$$

so that the Lagrangian in (77) can be written as

$$L = 3\sqrt[3]{W^\mu W_\mu}\,\sqrt{g} \tag{79}$$

The multi-momenta conjugate to c are

$$d^{*\alpha}_A \equiv \frac{\partial L}{\partial(\partial_\alpha c^A)} = (W^\nu W_\nu)^{-\frac{2}{3}}\sqrt{g}\,W_\mu \sigma^\mu_{A\dot{B}} g^{\alpha\beta}\partial_\beta c^{*\dot{B}} \tag{80}$$

which leads to the expressions

$$\frac{1}{2}g_{\alpha\beta}\, d^{*\alpha A} \bullet d^{\beta\dot{B}}\, g_{\gamma\delta} d^{\gamma *}_A \bullet d^\delta_{\dot{B}} = \sqrt[3]{W^\mu W_\mu}\, g^2 \tag{81}$$

$$d^{*\alpha}_A \bullet \partial_\alpha c^A + c.c. = 4\sqrt[3]{W^\mu W_\mu}\,\sqrt{g} \tag{82}$$

From (81), (82) and (79), we obtain the Dedonder-Weyl covariant Hamiltonian density

$$H(c,d) \equiv d^{*\alpha}_A \bullet \partial_\alpha c^A + c.c. - L = \frac{1}{2}g^{-\frac{3}{2}} g_{\alpha\beta}\, d^{*\alpha A} \bullet d^{\beta\dot{B}}\, g_{\gamma\delta}\, d^{*\gamma}_A \bullet d^\delta_{\dot{B}} \tag{83}$$

and hence the first order form

$$I = \int d^{*\alpha}_A \bullet \partial_\alpha c^A + c.c. - g^{-\frac{3}{2}}\, p^\mu p_\mu\, d\tau d\sigma\,, \quad p^\mu \equiv \frac{1}{2}\sigma^\mu_{A\dot{B}} g_{\alpha\beta}\, d^{*\alpha A} \bullet d^{\beta\dot{B}} \tag{84}$$

of the Polyakov action (77). Performing the substitution

$$d^{*\alpha}_A \to \sqrt{1-\phi^2}\, d^{*\alpha}_A \tag{85}$$

and using (76), this action can be written as

$$I = \int \sqrt{1-\phi^2}\big(d^{*\alpha}_A \bullet \partial_\alpha c^A + c.c. - h^{-\frac{3}{2}} p^\mu p_\mu\big)\, d\tau d\sigma \tag{86}$$

The scalar field ϕ corresponds to the coefficients ϕ_i in the discrete action (43), and the condition that they be constants of motion is replaced with the reparametrization invariant condition

$$\mu^\alpha \partial_\alpha \phi = 0 \tag{87}$$

where μ^α is the real vector density

$$\mu^\alpha \equiv \frac{1}{2}(d_A^{*\alpha} \bullet c^A + c.c.) \tag{88}$$

The equations of motion for the momenta are

$$\partial_\alpha d_A^{*\alpha} = 0 \tag{89}$$

which only determine them up to an arbitrary worldsheet scalar χ_A

$$d_A^{*\alpha} \to d_A^{*\alpha} + \epsilon^{\alpha\beta}\partial_\beta \chi_A^* \tag{90}$$

The class of solutions which correspond to the multi-particle model in section 5 is obtained by choosing χ_A so that $d_A^{*\beta}$ satisfies the reparametrization invariant condition

$$\mu^\alpha \epsilon_{\alpha\beta} d_A^{*\beta} = 0 \tag{91}$$

This condition will be imposed as a constraint on the action principle with Lagrange multipliers λ^A

$$\int \sqrt{1-\phi^2}\big(d_A^{*\alpha} \bullet \partial_\alpha c^A + c.c. - h^{-\frac{3}{2}} p^\mu p_\mu\big) - \lambda^A \bullet \mu^\alpha \epsilon_{\alpha\beta} d_A^{*\beta} + c.c.\, d\tau d\sigma \tag{92}$$

This action is invariant under a global $SL(2.\mathbb{C})$ and $U(1)$ gauge symmetry. As in the foregoing, the corresponding Noether currents are assumed to vanish

$$\mathcal{J}^\alpha_{\mathcal{AB}} \equiv \sqrt{1-\phi^2}(d_A^{*\alpha} \bullet c_B + d_B^{*\alpha} \bullet c_A) = 0\,,\, \jmath^\alpha \equiv i\sqrt{1-\phi^2}(d_A^{*\alpha} \bullet c^A - c.c.) = 0 \tag{93}$$

leading to the 'Noether condition'

$$d_A^{*\alpha} \bullet c^B = \mu^\alpha \delta_A^B \tag{94}$$

where μ^α is the real vector density (88). The equations of motion are obtained by independent variation of d and c

$$\partial_\alpha c^A = h^{-\frac{3}{2}} g_{\alpha\beta}\, p^{A\dot{B}} d^\beta_{\dot{B}} + \frac{1}{2} c^A(\lambda^B \bullet \epsilon_{\alpha\beta} d_B^{*\beta}) + \lambda^A \epsilon_{\beta\alpha}\mu^\beta \tag{95}$$

$$\partial_\alpha d_A^{*\alpha} = -\frac{1}{2} d_A^{*\beta} \lambda^B \bullet \epsilon_{\beta\gamma} d_B^{*\gamma} \tag{96}$$

where we have made use of (87). Taking the inner product of both sides of (95) with $c^{*\dot{B}}$ and contracting with μ^α, the λ-terms vanish and it reduces to

$$\mu^\alpha \partial_\alpha x^\mu = 2h^{-\frac{3}{2}} h_{\alpha\beta}\mu^\alpha \mu^\beta p^\mu \tag{97}$$

We observe that because of the constraint (91), the equations of motion determine only a single directional derivative of the space-time coordinates. This is

what we would expect for a multi-particle system. In a domain where μ^α is regular, μ^2 can be made to vanish through a reparametrization. Any subsequent reparametrization of the form $\tau \to \bar{\tau}(\tau,\sigma)$, $\sigma \to \overline{\sigma}(\sigma)$ preserves $\mu^2 = 0$ and can be chosen so that

$$h^{-\frac{3}{2}} h_{11} h_{11} = 1 \tag{98}$$

An easy way of seeing this is to consider the vector density

$$v^\gamma = h^{\frac{3}{8}} (h_{\alpha\beta}\mu^\alpha \mu^\beta)^{-\frac{1}{2}} \mu^\gamma \tag{99}$$

In the parametrization $\mu^2 = 0$, v^1 does not depend on μ^1. Since the weight of v is different from 1, there exists a reparametrization $\tau \to \bar{\tau}(\tau,\sigma)$, $\sigma \to \overline{\sigma}(\sigma)$ which preserves $\mu^2 = v^2 = 0$ and makes $v^1 = 1$. This gives (98). Note that in this parametrization, (87) says that ϕ is a constant of motion as in the discrete system.

When $\mu^2 = 0$, then according to (91), d^2 vanishes too. This makes the equations of motion (95) and (96) reduce to

$$\partial_1 c^A = \tilde{p}^{A\dot{B}} d^1_{\dot{B}}\,, \quad \tilde{p}_{A\dot{B}} \equiv d^{*1}_A \bullet d^1_{\dot{B}} \tag{100}$$

$$\partial_1 d^{*1}_A = 0 \tag{101}$$

which correspond to the action

$$I = \int \sqrt{1-\phi^2}\big(d^{*1}_A \bullet \partial_1 c^A + c.c. - \tilde{p}^\mu \tilde{p_\mu}\big)\, d\tau d\sigma \tag{102}$$

This action is the same as would have been obtained by inserting the coordinate conditions and the constraint (91) into the original action (86). It is of the same form as the multi-particle action (43) with summation replaced by integration.

8 The continuum limit

In a similar way as we did for the discrete system in section 5, we shall express the string action (102) as a particular instance of an explicit unitarily invariant action. Besides the metric δ_n, we shall also make use of the metric $\tilde{\delta}_n \equiv \delta_n(0)^{-1}\delta_n$ which converges to the one-point indicator function $I_{\{0\}}$.

Let $\overset{>}{C}$ and $\overset{<}{D}$ be vectors with components $c(\sigma)$ and $d^*(\sigma)$. In the discrete system, we started out from the assumption that the matrices P and Φ were diagonal. In the continuous system, this will be replaced by the limit conditions

$$\overset{>}{D}_{\dot{B}}\,(\tau,\sigma) \bullet \overset{<}{D}_A\,(\tau,\sigma') \to d_{\dot{B}}(\tau,\sigma) \bullet d^*_A(\tau,\sigma)\tilde{\delta}_n(\sigma-\sigma') \text{ for } n \to \infty \tag{103}$$

$$\Phi(\sigma,\sigma') \to \sqrt{1-\phi(\sigma)^2}\,\delta_n(\sigma-\sigma') \text{ for } n \to \infty \tag{104}$$

In a pseudo-unitarily invariant system, all contractions such as matrix multiplication and taking the trace must be performed with the metric δ_n. It follows that the action

$$I = \int Tr\left(\delta_n \Phi \delta_n \left(\frac{d}{d\tau}\overset{>}{C}^A \bullet \overset{<}{D}_A + h.c. - \delta_n(0) H\right)\right) d\tau \tag{105}$$

$$P_{A\dot{B}}(\tau,\sigma,\sigma') \equiv \overset{>}{D}_{\dot{B}}(\tau,\sigma) \bullet \overset{<}{D}_A(\tau,\sigma')\,, \quad H \equiv P^\mu \delta_n P_\mu \tag{106}$$

is preserved by the transformations (45),(41) with U being a pseudo-unitary matrix satisfying (71). In the continuum limit, this pseudo-unitary symmetry becomes a unitary one. Upon inserting the expressions (103) and (104) into the action (105), we find that it converges to the string action (102) in the continuum limit $n \to \infty$. Corresponding to (55) in the discrete system, the action (105) yields the 'Noether condition'

$$\overset{>}{C}^A(\tau,\sigma) \bullet \overset{<}{D}_B(\tau,\sigma') = M(\tau,\sigma,\sigma')\,\delta^A_B \tag{107}$$

where $M(\tau)$ is a Hermitian matrix. This condition becomes pseudo-unitarily invariant by choosing M to be

$$M(\tau,\sigma,\sigma') = \mu^1(\tau)\tilde{\delta}_n(\sigma - \sigma') \tag{108}$$

Since $\tilde{\delta}_n(0) = 1$, the 'Noether condition' (94) for the string is recovered by setting $\sigma = \sigma'$. The canonical equations of motion are derived from the action (105) by independent variation of $\overset{>}{C}$ and $\overset{<}{D}$

$$\frac{d\overset{>}{C}^A}{d\tau} = \delta_n(0)\frac{\partial H}{\partial P_{A\dot{E}}}\overset{>}{D}_{\dot{E}}\,, \quad \frac{d\overset{<}{D}_A}{d\tau} = -\delta_n(0)\,\overset{<}{C}^{\dot{E}}\,\frac{\partial H}{\partial X^{A\dot{E}}}\,(=0) \tag{109}$$

Taking the inner product of both sides of these equations with $\overset{<}{C}^{\dot{B}}$ and $\overset{>}{D}_{\dot{B}}$, and applying (108), we obtain

$$\frac{dX^\mu}{d\tau} = \mu^1(\tau)\tilde{\delta}_n\delta_n(0)\frac{\partial H}{\partial P_\mu}\,, \quad \frac{dP_\mu}{d\tau} = -\mu^1(\tau)\tilde{\delta}_n\delta_n(0)\frac{\partial H}{\partial X^\mu}\,(=0) \tag{110}$$

Since $\tilde{\delta}_n\delta_n(0) = \delta_n$, then after a reparametrization $\frac{d\bar{\tau}}{d\tau} = 2m\mu^1(\tau)$, these equations converge to the space-time canonical equations of motion

$$\frac{dX^\mu}{d\bar{\tau}} = \frac{\partial \mathcal{H}}{\partial P_\mu}\,, \quad \frac{dP_\mu}{d\bar{\tau}} = -\frac{\partial \mathcal{H}}{\partial X^\mu}\,(=0)\,, \quad \mathcal{H} = \frac{1}{2m}P^\mu P_\mu \tag{111}$$

in the continuum limit $n \to \infty$. From this point on, the discussion is the same as in the discrete case.

9 Concluding remarks

We have shown that the quantized free relativistic point particle is equivalent to a particular parametrization of a classical relativistic string in Clifford Space. In obtaining this result, the solutions to the equations of motion were restricted to those which satisfied (91). It remains to determine the physical content of the full dynamics of the string.

There are good reasons to believe that a $1 + 3$ dimensional Space-Time does not suffice to accommodate the particle physics of the Standard Model. The Clifford model is limited to an $SO(1.3)$ Space-Time because it is based on complex-valued Weyl spinors. Since they are an integral part of the model, it is difficult to see how the dimension of Space-Time can be increased without resorting to Quaternion- or Octonion-valued spinors. For algebraic reasons [3], such spinors would be expected to generate an $SO(1.5)$ and an $SO(1.9)$ Space-Time respectively.

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